Notation & "Pre-Req" material

- B is unit Euclidean ball.
- $C_i + \varepsilon B = \{x \mid |x-y| < \varepsilon$ for at least one $y \in C_i\}$.
- associated $\longrightarrow$ assoc.
- half-space $\longrightarrow$ h.s.

Section 10 (Continuity of Convex Functions)

Def'n

A function f on $\mathbb{R}^n$ is said to be <u>continuous relative to a subset S</u> of $\mathbb{R}^n$ if the restriction of f to S is a continuous function.

That is, for $x \in S$ $f(y) \to f(x)$ as $y \to x$ inside S, but necessarily as $y \to x$ from outside S.

Theorem 10.1

A convex function f on $\mathbb{R}^n$ is continuous relative to any relatively open convex set C in it's effective domain, in particular relative to $ri(dom(f))$.

Proof

Define
$$g(x) = \begin{cases} f(x) & \text{if } x \in C \\ +\infty & \text{if } x \notin C \end{cases}.$$
Then $dom(g) = C$

Therefore WLOG we assume $C = dom(f)$ (otherwise we simply replace f w/ g). Also WLOG assume C is n-dimensional (Then C is indeed open rather than simply relatively open).

If f is improper, then $f \equiv -\infty$ on C (Thm 7.2: Improper cvx f is identically $-\infty$ for $x \in ri(dom\, f)$)

and continuity is trivial, therefore assume f is proper. Since f is cvx & proper and C is open Thm 7.4 $\Rightarrow$ $\mathrm{cl} f = f$ i.e. f is lower semi-continuous. (Left to show f is upper semi-continuous which from Thm 7.1 it suffices to show the level sets $\{x \mid f(x) \geq \alpha\}$ are all closed $\forall\, \alpha \in \mathbb{R}$).

Since $C = \mathrm{dom} f$ is open, Lemma 7.3 implies

$$\mathrm{int}(\mathrm{epi}\, f) = \{(x,\mu) \mid \mu > f(x),\ \mu \in \mathbb{R}\}.$$

Consider $\mathrm{int}(\mathrm{epi}\, f) \cap \{(x,\mu) \mid \mu < \alpha\} \in \mathbb{R}^{n+1}$ which is open, when projected onto $\mathbb{R}^n$ this is $\{x \mid f(x) < \alpha\}$ which is therefore also open implying its complement $\{x \mid f(x) \geq \alpha\}$ is closed. $\square$

Corollary 10.1.1

A convex function which is finite on all of $\mathbb{R}^n$ is necessarily continuous.

Example

Let $\mathfrak{F}(x,t)$ be a real-valued fct on $\mathbb{R}^n \times T$ (w/ T an arbitrary set), s.t. f is cvx as a fct of x for each t & bounded above as a fct of t for each x. (This situation would arise, say, if one had a finite convex fct on $\mathbb{R}^n$ continuously depent on the time t over a certain closed interval T.) Then

$$h(x) = \sup\{\mathfrak{F}(x,t) \mid t \in T\}$$

depends continuously on x.

Proof

h is cvx since it is the pointwise supremum of cvx fcts $\mathfrak{F}(x,t)$ bounded above for each x
$\Rightarrow$ $h(x)$ is finite $\forall\ x \in \mathbb{R}^n$. Then corollary 10.1.1 $\Rightarrow$
$h(x)$ is continuous.